

THE NODAL COUNT $\{0, 1, 2, 3, \dots\}$ IMPLIES THE GRAPH IS A TREE

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ABSTRACT. Sturm's oscillation theorem states that the n^{th} eigenfunction of a Sturm-Liouville operator on the interval has $n - 1$ zeros (nodes) [57, 58]. Recent works generalized this result for all metric tree graphs [51, 54] and Fiedler proved a similar result for discrete tree graphs [28]. We prove the converse theorems for both discrete and metric graphs. Namely, if for all n , the n^{th} eigenfunction of the graph has $n - 1$ zeros then the graph is a tree. The proof also shows that when the graph is supplied with a magnetic field it is not possible for all (or even almost all, in the metric case) the eigenvalues to exhibit a diamagnetic behaviour. In addition, we develop a notion of 'discretized' versions of a metric graph and show that their nodal counts are strongly connected to this of the metric graph.

1. INTRODUCTION

Nodal domains were first presented in full glory by Chladni's Sound figures. By the end of the 18th century Chladni was performing the following demonstration: he spread sand on a brass plate and stroke it with a violin bow. This caused the sand to accumulate in intricate patterns of nodal lines - the lines where the vibration amplitude vanishes. The areas bounded by the nodal lines are the nodal domains. The first rigorous result on nodal domains is probably Sturm's oscillation theorem, according to which a vibrating string is divided into exactly n nodal intervals by the zeros of its n^{th} vibrational mode [57, 58] (and see also [22], p. 454). In the next century Courant treated vibrating membranes and had proved that the number of nodal domains of the n^{th} eigenfunction of the Laplacian is bounded from above by n [21] (see also [22], p. 452). Pleijel further restricted the possible nodal domain counts and showed for example, that the Courant bound can be attained only a finite number of times [50]. These are some of the earlier results in the field of counting nodal domains. This field had gained an exciting turn when Blum, Gnutzmann and Smilansky have shown that the nodal count statistics may reveal the nature of the underlying manifold - whether its classical dynamics is integrable or chaotic, [16]. This opened a new research direction of treating the nodal count from the inverse problems perspective. One aspect of this research is rephrasing the famous question of Mark Kac by asking 'Can one *count* the shape of a drum?' (see [38] for Kac's original question). Namely, what can one learn about an object (manifold, graph, etc.), knowing the nodal counts of all of its eigenfunctions? A possible way to treat this question is by studying the direct (rather than inverse) problem and developing formulae which describe nodal count sequences ([1, 2, 5, 32, 39, 40]). Such formulae are expected to reveal various geometric properties of the underlying object which one seeks to obtain. One can also study inverse nodal problems by comparing the nodal information with the spectral one. A well established conjecture in the field claims that isospectral objects have different nodal count sequences. After its first appearance in a paper by Gnutzmann, Smilansky and Sondergard [34], the conjecture initiated a series of works on nodal counts of various objects, either affirming the conjecture in certain settings [6, 7, 8, 18, 47], or pointing on counter examples [17, 48]. The general validity of this conjecture is still not well understood. The most recent approach in the study of nodal counts tries to describe the nodal domain count (and even morphology) of individual eigenfunctions. This was triggered by Helffer, Hoffmann-Ostenhof and Terracini who study Schrödinger operators on two-dimensional domains using partitions of the domain, [37]. They provide characterization of the

morphology and number of nodal domains of eigenfunctions which attain the Courant bound. A later work used their partition approach for metric graphs and enabled the characterization of the number of nodal points and their location for all eigenfunctions via a Morse index of an energy function, [4]. This result initiated a series of works which provide similar connections between nodal counts and Morse indices, for discrete graphs and manifolds as well, [13, 31] and [11, 26, 14]. The three latter works concern the response of an eigenvalue to magnetic fields on the graph and its relation to the nodal count (the last of them even appears in this collection).

The current paper adopts this magnetic approach to solve inverse nodal problems on both metric and discrete graphs. For metric graphs, 'nodal lines' are actually nodal points, the zeros of the eigenfunction, and the nodal domains are the subgraphs bounded in between. For discrete graphs, 'nodal lines' are edges connecting two vertices at which the eigenvector differs in sign, and the nodal domains are the subgraphs obtained upon removal of those edges. The main result of this paper is a solution to an inverse nodal problem - under some genericity assumptions, if for all n the n^{th} eigenfunction has $n - 1$ zeros, then the graph cannot have cycles and must be a tree. This result is valid for both metric and discrete graphs, although the proof methods are different. We conclude this introductory part by referring the reader to the collection of articles, [56], where a broad view is given on the nodal domain research, its history, applications and the numerous types of objects it concerns.

The paper is set in the following way. In the rest of the introductory section we familiarize the reader with all ingredients which will play a role in the formulation of the theorems: magnetic operators on both discrete and metric graphs, the connection between both types of graphs and their nodal counts. Section 2 introduces the three main tools needed for the proofs. The next two sections bring the proofs of our inverse nodal theorems for both discrete and metric graphs, each in a separate section. Section 5 presents a method of discretizing metric graphs which gives another insight for the proof in section 4 and might lead to generalizations of the inverse result. Finally we discuss the essence of the work and offer further exploration directions. It is important to emphasize that some of the paper's sections can be read independently of those preceding them. The schematic diagram in figure 1.1 shows the section dependencies as blocks resting on top of other blocks on which they depend.

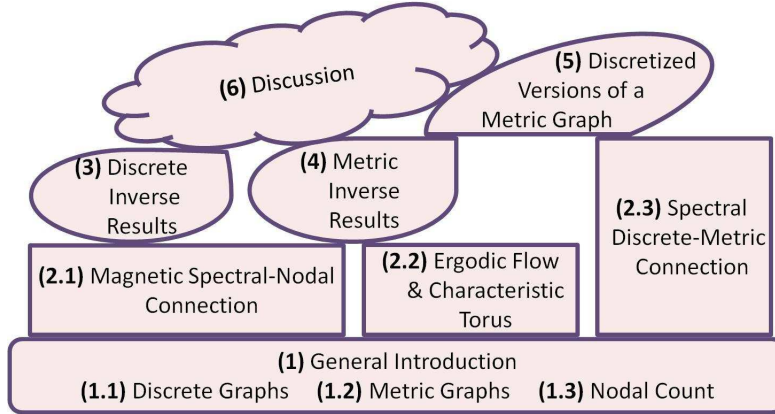


FIGURE 1.1. Schematic diagram of section dependencies

1.1. Discrete graphs . Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph with the sets of vertices \mathcal{V} and edges \mathcal{E} . All graphs discussed in this paper are connected and have a finite number of vertices and edges. Each edge $e \in \mathcal{E}$, $e = \{u, v\}$ connects some two vertices, $u, v \in \mathcal{V}$. We exclude edges which connect a vertex to itself and do not allow two vertices to be connected by more than one

edge. For some purposes we would need to consider the directions of the edge. We therefore also define the set $\overleftrightarrow{\mathcal{E}}$ which contains all edges of \mathcal{E} and each of them with its both directional versions (and therefore $|\overleftrightarrow{\mathcal{E}}| = 2|\mathcal{E}|$). We use the notation $e = (u, v)$ to refer to a directed edge $e \in \overleftrightarrow{\mathcal{E}}$ which starts at u and terminates at v and denote the edge with the reversed direction by $\hat{e} := (v, u)$. For a vertex $v \in \mathcal{V}$, its *degree* equals the number of edges connected to it, i.e., $d_v := |\{u; \{u, v\} \in \mathcal{E}\}|$. *Functions (vectors)* on the graph refer to $f : \mathcal{V} \rightarrow \mathbb{R}$ and we now present the operators acting on such functions.

The *normalized Laplacian* is

$$(1.1) \quad \mathbf{L}_{u,v}^{(norm)} = \begin{cases} -1/\sqrt{d_u d_v} & (v, u) \in \overleftrightarrow{\mathcal{E}} \\ 0 & (v, u) \notin \overleftrightarrow{\mathcal{E}} \\ 1 & u = v \end{cases}.$$

The normalized Laplacian is a special case of the *generalized discrete Laplacian* which we call *discrete Schrödinger operator*, a real symmetric matrix which obeys

$$(1.2) \quad \begin{aligned} \mathbf{L}_{u,v} &= \mathbf{L}_{v,u} < 0 \text{ if } (v, u) \in \overleftrightarrow{\mathcal{E}} \\ \mathbf{L}_{u,v} &= 0 \text{ if } (v, u) \notin \overleftrightarrow{\mathcal{E}}, \end{aligned}$$

and with no constraints on its diagonal values (which are sometimes called in the literature on site potentials).

These matrices have real eigenvalues which we denote by $\{\lambda_n\}_{n=1}^{|\mathcal{V}|}$, ordered increasingly, and the corresponding eigenvectors are denoted by $\{f_n\}_{n=1}^{|\mathcal{V}|}$. The spectrum of the normalized Laplacian belongs to the interval $[0, 2]$. We refer the interested reader to many results concerning the spectra of graph matrices and their eigenvectors which can be found in the books [15, 20, 23, 24] and the references within.

The discrete Schrödinger operator can be supplied with a *magnetic potential*, \mathcal{A} , which is defined as $\mathcal{A} : \overleftrightarrow{\mathcal{E}} \rightarrow \mathbb{R}$ such that $\mathcal{A}(e) = -\mathcal{A}(\hat{e})$. Assigning such a potential to the operator amounts to the following changes

$$\mathbf{L}(\mathcal{A}) : \mathbb{C}^{|\mathcal{V}|} \rightarrow \mathbb{C}^{|\mathcal{V}|}$$

$$\mathbf{L}_{u,v}(\mathcal{A}) = \mathbf{L}_{u,v} e^{i\mathcal{A}(u,v)},$$

where $\mathbf{L}_{u,v}$ are the entries of the previous (zero magnetic potential) Laplacian. Given a cycle, $\gamma = (v_1, \dots, v_n)$, on the graph we define the *magnetic flux* through this cycle as

$$\alpha_\gamma := [\mathcal{A}(v_1, v_2) + \dots + \mathcal{A}(v_{n-1}, v_n) + \mathcal{A}(v_n, v_1)] \bmod 2\pi.$$

The graph's cycles play an important role when introducing magnetic potential. We denote the number of “independent” cycles on the graph as $\beta := |\mathcal{E}| - |\mathcal{V}| + 1$ (assuming the graph is connected). This is also known as the first Betti number, which is the dimension of the graph's first homology. We will use this throughout the paper by fixing some β independent cycles on the graph, $\{\gamma_1, \dots, \gamma_\beta\}$ and denoting the magnetic fluxes through them by $\{\alpha_1, \dots, \alpha_\beta\}$. One can show that given two magnetic potentials, \mathcal{A}_1 and \mathcal{A}_2 with the same values for the magnetic fluxes $\{\alpha_i\}_{i=1}^\beta$, their corresponding magnetic Laplacians, $\mathbf{L}(\mathcal{A}_1)$ and $\mathbf{L}(\mathcal{A}_2)$ are unitarily equivalent. This unitary equivalence is also known as the gauge invariance principle and means that the spectrum of the Laplacian is uniquely determined by the values $\{\alpha_i\}_{i=1}^\beta$ (but not so for its eigenfunctions). We will take advantage of this principle, notation-wise, and from now on we will write the Laplacian and its eigenvalues as functions of the magnetic parameters (fluxes), $\mathbf{L}(\vec{\alpha})$ and $\lambda_n(\vec{\alpha})$, where $\vec{\alpha} := (\alpha_1, \dots, \alpha_\beta)$. Note that in the special case of a tree graph, the gauge invariance principle means that all magnetic Laplacians are unitarily equivalent to the zero magnetic Laplacian, $\mathbf{L}(0)$, and the eigenvalues do not depend on the magnetic potential. More details on magnetic operators on discrete graphs can be found in [27, 46, 55, 59].

1.2. Metric (quantum) graphs . A *metric graph* is a discrete graph each of whose edges, $e \in \mathcal{E}$, is identified with a one dimensional interval, $[0, l_e]$, with a positive finite length l_e . We use the notation \vec{l} for the vector whose entries are $\{l_e\}_{e \in \mathcal{E}}$. We denote a metric graph by $\Gamma = (\mathcal{V}, \mathcal{E}, \vec{l})$. We can then assign to each edge $e \in \overleftrightarrow{\mathcal{E}}$ a coordinate, x_e , which measures the distance along the edge from the starting vertex of e . In particular, we have the following relation between coordinates of reversed edges, $x_e + x_{\hat{e}} = l_e$. We denote a coordinate by x , when its precise nature is unimportant.

We equip the metric graphs with a self-adjoint differential operator \mathcal{H} , the *Hamiltonian* or *metric Schrödinger operator*,

$$(1.3) \quad \mathcal{H}(\mathcal{A}) : f_e(x_e) \mapsto \left(i \frac{d}{dx_e} + \mathcal{A}_e(x_e) \right)^2 f_e(x_e) + V_e(x_e) f_e(x_e) .$$

where $V_e(x)$ is a real valued bounded and piecewise continuous function which forms the *electric potential*, and $\mathcal{A}_e \in C^1([0, l_e])$ is called the *magnetic potential* and it obeys $\mathcal{A}_e(x_e) = -\mathcal{A}_{\hat{e}}(x_{\hat{e}})$. It is most common to call this setting of a metric graph with a Schrödinger operator, a quantum graph. We will keep calling these graphs metric graphs to distinguish them from their discrete counterpart which we also equip here with an operator of a quantum nature.

To complete the definition of the operator we need to specify its domain. We denote by $H^2(\Gamma)$ the following direct sum of Sobolev spaces

$$(1.4) \quad H^2(\Gamma) := \bigoplus_{e \in \overleftrightarrow{\mathcal{E}}} H^2([0, l_e]) ,$$

where we denote by f_e the restriction of f to the edge e and require $f_e(x_e) = f_{\hat{e}}(l_e - x_e)$, for all $e \in \overleftrightarrow{\mathcal{E}}$. Therefore, in affect, there is only a single function assigned to each pair of edges which are the reverse of each other, and we may denote it by $f_e(x_e)$ with $e \in \mathcal{E}$.

All conditions matching functions at the vertices that lead to the operator (1.3) being self-adjoint have been classified in [36, 41, 44]. It can be shown that under these conditions the spectrum of \mathcal{H} is real and bounded from below [44]. In addition, since we only consider compact graphs, the spectrum is discrete and with no accumulation points. We number the eigenvalues in the ascending order and denote them (similarly to the discrete case) with $\{\lambda_n\}_{n=1}^{\infty}$ and their corresponding eigenfunctions with $\{f_n\}_{n=1}^{\infty}$. We also use k_n , such that $\lambda_n = k_n^2$, and say that $\{k_n\}_{n=1}^{\infty}$ is the k -spectrum of the graph.

As we wish to study the sign changes of the eigenfunctions, we would require their continuity. The only matching conditions that guarantee that the function is continuous (at the vertices) is the so-called extended δ -type conditions at all the graph's vertices.

A function $f \in H^2(\Gamma)$ is said to satisfy the extended δ -type conditions at a vertex v if

- (1) f is continuous at $v \in \mathcal{V}$, i.e.,

$$\forall e_1, e_2 \in \overleftrightarrow{\mathcal{E}}_v \quad f_{e_1}(0) = f_{e_2}(0),$$

where $\overleftrightarrow{\mathcal{E}}_v$ is the set of edges starting at v .

- (2) the outgoing derivatives of f at v satisfy

$$(1.5) \quad \sum_{e \in \overleftrightarrow{\mathcal{E}}_v} \left(\frac{d}{dx_e} - i \mathcal{A}_e(0) \right) f_e(0) = \chi_v f(0), \quad \chi_v \in \mathbb{R},$$

where $f(0)$ denotes the value of f at the vertex (which is uniquely defined due to the first part of the condition).

In particular, the case $\chi_v = 0$ is often referred to as *Neumann condition* (also called Kirchhoff or standard condition). We will call a graph whose vertex conditions are all of Neumann type and whose magnetic and electric potentials vanish everywhere, a *Neumann graph* and state our main results for such graphs. A Neumann graph has $\lambda_1 = 0$ with multiplicity which equals

the number of graph's components (which is always one throughout this paper) and their k -spectrum is therefore real and positive. Another useful vertex condition is $\forall e \in \overleftrightarrow{\mathcal{E}}_v, f_e(0) = 0$. This is called a *Dirichlet condition* at the vertex v , and can be formally written as (1.5) with $\chi_v = \infty$. Note that whenever a vertex exhibits a Dirichlet condition, it effectively disconnects all edges connected to this vertex. Similarly to the Neumann graph, a *Dirichlet graph* is obtained whenever all vertices are supplied with Dirichlet conditions. The eigenvalues of such a graph are merely a union of spectra of its disjoint edges (with Dirichlet conditions at their endpoints) and are called *Dirichlet eigenvalues*.

An important observation which plays a role in the paper is that the spectrum and eigenfunctions of the graph are not affected if a graph's edge is divided into two parts by introducing a new vertex (of degree two) at an arbitrary point on this edge and supplying it with Neumann conditions. We call the process (and outcome) of introduction of any number of such vertices on the graph, graph's *subdivision* and keep in mind that spectral properties are invariant to subdivision. Similarly to the definition of discrete graphs, we also exclude edges which connect a vertex to itself and vertices which are connected by more than two edges (reversed to each other). Note, however, that this does not restrict the generality of our results as any metric graph which does have self cycles or multiple edge can be subdivided to eliminate these defects.

The *magnetic flux* through a cycle, $\gamma = (v_1, \dots, v_n)$, is defined as

$$\alpha_\gamma := \sum_{e \in \gamma} \int_0^{l_e} \mathcal{A}_e dx_e,$$

where the notation $e \in \gamma$ means that either $e = (v_i, v_{i+1})$ for some i or $e = (v_n, v_1)$. The gauge invariance principle introduced previously applies to the metric Schrödinger operator as well and allows us to write $\mathcal{H}(\vec{\alpha})$ and $\lambda_n(\vec{\alpha})$, when referring to the operator or its eigenvalues. This obviously holds once some fixed choice β cycles is made, and the notation $\{\alpha_i\}_{i=1}^\beta$ is adapted to fluxes through these cycles, where $\vec{\alpha} := (\alpha_1, \dots, \alpha_\beta)$. Two good references for further reading on the general theory of metric (quantum) graphs are [12, 33].

1.3. The nodal count. The main focus of this paper is the number of sign changes of eigenfunctions on discrete and metric graphs. For counting sign changes we always consider eigenfunctions of the zero magnetic potential operators, as otherwise we are not guaranteed to have real valued eigenfunctions. In addition, in order for the number of sign changes to be well defined we need the following assumption for eigenfunctions of both metric and discrete graphs.

Assumption 1. *The eigenvalue $\lambda_n(\vec{0})$ is simple and the corresponding eigenfunction, $f_n(\vec{0})$ is different than zero on every vertex.*

We call an eigenvalue generic if it satisfies this assumption and throughout the paper we sometimes adopt this assumption for a specific eigenvalue and in other cases we place this requirement on the whole spectrum. This assumption is generic with respect to various perturbations to the operator, some of which are discussed in [11, 4, 30].

The definition of sign changes differs from metric to discrete graphs and we start by presenting the former case. Let $f_n \in H^2(\Gamma)$ be the n^{th} eigenfunction of the Schrödinger operator on a metric graph, such that assumption 1 holds for it. The zeros of f_n form isolated points on the graph - they correspond to the function's sign changes and their number is called the *sign change count* or *nodal point count* of f_n and denoted by ϕ_n . If one removes the nodal points of f_n from the graph Γ , it separates into subgraphs disconnected one from the other. These subgraphs are called the *nodal domains* of f_n and their number, the *nodal domain count*, is denoted by ν_n . Turning to the discrete case, the situation is somewhat different, as due to assumption 1 a function $f_n \in \mathbb{R}^\mathcal{V}$ does not vanish at all. We then say that an edge $e = (u, v)$ forms a *sign change (also nodal point)* of the eigenfunction if $f_n(u)f_n(v) < 0$. The total number of nodal points is denoted by $\phi_n := |\{(u, v) \in \mathcal{E} \mid f_n(u)f_n(v) < 0\}|$. Once again, removing the edges

which form nodal points separates the graph into subgraphs which we call nodal domains and all the terminology and notations are the same as in the metric case. In both the discrete and the metric cases we focus our attention to the sequences of the counts, $\{\phi_n\}$ and $\{\nu_n\}$, which are finite (of size $|\mathcal{V}|$) in the discrete case and infinite in the metric one. We will abuse the terms and refer to the sequences $\{\phi_n\}$ and $\{\nu_n\}$ as the nodal point count and the nodal domain count, just as we call the individual numbers. In the current paper we mainly treat the nodal point count of graphs and give only brief remarks regarding the nodal domain count. We might also use the term *nodal count*, when it is either clear or unimportant whether we count nodal points or nodal domains. We end this definitions paragraph by adding that if assumption 1 is not satisfied since the eigenfunction has zero entries, the definitions above should be modified. There are indeed alternative definitions of the nodal count in such scenarios (see [6]), but their treatment is out of the scope of this paper.

Most of the known results regarding nodal counts on graphs are in the form of bounds on these numbers. Interestingly, these bounds apply both for discrete and metric graphs (the generality assumption 1 is assumed throughout). The bounds on the nodal point count are

$$(1.6) \quad n - 1 \leq \phi_n \leq n - 1 + \beta.$$

For discrete graphs, these bounds hold for eigenfunctions of the generalized Laplacian, (1.2), and were proven in [10, 13]. For metric graphs, the bounds hold for eigenfunctions of the Schrödinger operator, (1.3), (with zero magnetic potential) and proved in [4, 10].

In particular, for a tree graph where $\beta = 0$, one obtains $\forall n \phi_n = n - 1$. Therefore, a *metric* tree graph shares the same nodal count with the interval. This result for the interval is the famous Sturm's oscillation theorem [57, 58] and its generalization for trees was done in [51, 54]. The fact that *discrete* tree graphs also have this nodal count was proved by Fiedler [28]. In the current paper we prove the converse statements. Namely, under the genericity assumption, if $\forall n \phi_n = n - 1$ then the graph must be a tree graph. We prove this both for discrete graphs and for Neumann metric graphs. It turns out to be useful to define the difference of the nodal count from its value in the tree case

$$(1.7) \quad \sigma_n := \phi_n - n + 1.$$

We call this quantity *nodal surplus* following [11] (it was also called *nodal defect* in [26]).

Similar bounds to (1.6) exist for the nodal domain count, as well

$$(1.8) \quad n - \beta \leq \nu_n \leq n.$$

The proofs for metric graphs are found in the same papers as the nodal point count bounds, [4, 10]. For discrete graphs, the upper bound (Courant bound) is proved in [25] and the lower in [10]. Two recent works which go further than the above bounds (but from which the bounds (1.6), (1.8) can be deduced) characterize the nodal count of an eigenfunction in terms of a Morse index of a predefined energy function, [4, 13]. These works led to a characterization of the nodal count in terms of Morse indices of magnetic perturbations. This is a main tool used in the proofs of the current paper and is described in the next section.

2. INTRODUCING TOOLS NEEDED FOR THE PROOFS

This section introduces a few tools which are useful for the proofs in the paper. Most of this section's content appeared in previous works. We bring it here in a format and notations which enables its usage in the rest of the paper and allow ourselves to perform necessary modifications and make small additions.

2.1. The magnetic spectral-nodal connection. This subsection is devoted to an important connection between the nodal count and stability of eigenvalues under magnetic perturbation. Such a connection first appeared in [11], where Berkolaiko proved it for discrete graphs. Shortly afterward, the same theorem was reproved by Colin de Verdière who also had shown that the

theorem holds for the Hill operator - the metric Schrödinger operator on a single circle graph, [26]. The proof of the theorem for a general metric graph, due to Berkolaiko and Weyand, appears in another manuscript of this same issue, [14].

The theorem forms an important tool in the proofs of this paper for both discrete and metric graphs and we indeed cite it here in a version which fits both cases

Theorem 1. [11, 14, 26] *Let $\mathcal{G}(\Gamma)$ be a discrete (metric) graph supplied with a discrete (metric) Schrödinger operator. Let $\lambda_n(\vec{\alpha})$ and $f_n(\vec{\alpha})$ be an eigenvalue and a corresponding eigenfunction of the magnetic operator such that $\lambda_n(\vec{0})$ and $f_n(\vec{0})$ satisfy assumption 1. Then the following holds*

- (1) *The point $\vec{\alpha} = \vec{0}$ is a critical point of the function $\lambda_n(\vec{\alpha})$.*
- (2) *The critical point $\vec{\alpha} = \vec{0}$ is non-degenerate.*
- (3) *The nodal surplus, σ_n , of the eigenfunction $f_n(\vec{0})$ is equal to the Morse index of this critical point — the number of negative eigenvalues of the Hessian of $\lambda_n(\vec{\alpha})$ at $\vec{\alpha} = \vec{0}$. Namely,*

$$\sigma_n = \mathcal{M}_{\lambda_n}(\vec{0}).$$

2.2. Ergodic flow on the characteristic torus. It is well known that eigenvalues of a metric graph are given as zeros of a secular function [43, 42],

$$\left\{ k^2; \tilde{F}(k; \vec{l}; \vec{\alpha}) = 0 \right\}$$

$$(2.1) \quad \tilde{F}(k; \vec{l}; \vec{\alpha}) := \det \left(e^{-i/2(\mathbf{A}(\vec{\alpha}) + k\mathbf{E}(\vec{l}))} \right) \det(\mathbf{S}(k))^{-1/2} \det \left(\mathbf{1} - e^{i(\mathbf{A}(\vec{\alpha}) + k\mathbf{E}(\vec{l}))} \mathbf{S}(k) \right),$$

where \mathbf{A} , \mathbf{E} and \mathbf{S} are square matrices of dimension $|\vec{\mathcal{E}}|$ and contain the information about the magnetic fluxes, edge lengths and edge connectivity, respectively. Exact details on the structure of those matrices appear in [33, 43], and we just state here the necessary facts which we use later on

- (1) \mathbf{E} is a diagonal matrix of the form

$$\mathbf{E} = \text{diag} \{ l_e \}_{e \in \vec{\mathcal{E}}}$$

- (2) For a Neumann graph \mathbf{S} is a constant unitary matrix which does not depend on k .
- (3) \mathbf{A} is a diagonal matrix which is linear in $\vec{\alpha}$.
- (4) \tilde{F} is a real valued function.

Let $\Gamma = (\mathcal{V}, \mathcal{E}, \vec{l})$ be a metric graph. We write its edge lengths as the following linear combination

$$(2.2) \quad \forall e \in \mathcal{E} \quad l_e = \sum_{i \in I} r_i^{(e)} \xi_i,$$

where all $r_i^{(e)}$ are rational numbers and $\{\xi_i\}_{i \in I}$ is a set of *incommensurate* real numbers, i.e. they are linearly independent over the rationals. For example, if the edge lengths are incommensurate themselves then $|I| = |\mathcal{E}|$ and the set $\{\xi_i\}_{i \in I}$ can be chosen to be equal the edge lengths. On the other extreme, if all ratios of edge lengths are rational, then $|I| = 1$ and one can choose ξ_1 to equal any of the edge lengths (or any rational multiple of it). The relations between the edge lengths of the graph, $\{l_e\}_{e \in \mathcal{E}}$, and the parameters $\{\xi_i\}_{i \in I}$ will play an important role later on and we thus define the *length map* as

$$\mathcal{L} : \mathbb{R}^I \rightarrow \mathbb{R}^{\mathcal{E}}$$

$$(2.3) \quad [\mathcal{L}(\vec{x})]_e = \sum_{i \in I} r_i^{(e)} x_i.$$

This map depends on the specific edge lengths of Γ (even the dimension of its domain depends on that) and on the specific choice of values for $\{\xi_i\}_{i \in I}$.

We now describe a method introduced by Barra and Gaspard [9] who related the graph eigenvalues to the Poincaré return times of a flow to a hyperplane defined by the zero level set of the secular function (2.1). We present their method using our length map and start by redefining the secular function

$$(2.4) \quad F : \mathbb{R}^I \times \mathbb{R}^\beta \rightarrow \mathbb{R}$$

$$(2.5) \quad F(\vec{x}; \vec{\alpha}) := \tilde{F}(1; \mathcal{L}(\vec{x}); \vec{\alpha}).$$

We point on some properties of F

- (1) F is differentiable.
- (2) $F(k\vec{\xi}; \vec{\alpha}) = \tilde{F}(k; \mathcal{L}(\vec{\xi}); \vec{\alpha})$ since \mathcal{L} is homogeneous and \tilde{F} contains the parameter k only in the product $k\vec{l} = k\mathcal{L}(\vec{\xi})$.
- (3) $F(\vec{x}; \vec{\alpha})$ is periodic in each of the entries of \vec{x} . The period depends on the specific entry, x_i , and is some rational multiple of 2π .

The last property allows us to define the function $F(\cdot; \vec{\alpha})$ on an $|I|$ -dimensional torus, $\mathbb{T}^{|I|}$, with sides depending on the periodicity of F with respect to its parameters. $\{x_i\}_{i \in I}$. Namely,

$$F : \mathbb{T}^{|I|} \times \mathbb{R}^\beta \rightarrow \mathbb{R},$$

and from now on whenever F is mentioned, its \vec{x} variable is taken modulus this torus periodicity, even if this is not explicitly written. Property (2) allows to characterize the graph k -eigenvalues as

$$\left\{ k(\vec{\alpha}); F(k\vec{\xi}; \vec{\alpha}) = 0 \right\},$$

where $\vec{\xi} = \mathcal{L}^{-1}(\vec{l})$ is the vector with incommensurate entries chosen above. We therefore define the following flow on the torus

$$(2.6) \quad \vec{x}(k) := k\vec{\xi},$$

and the hyperplane

$$\Sigma_{\vec{\alpha}} := \left\{ F(k\vec{\xi}; \vec{\alpha}) = 0 \right\},$$

so that the k -spectrum is obtained as the times (i.e., the k values) for which the flow $\vec{x}(k)$ pierces $\Sigma_{\vec{\alpha}}$.

Remark 2. The eigenfunctions which correspond to the graph eigenvalues depend both on the point on the torus, $k\vec{\xi} \in \mathbb{T}^{|I|}$, and on the specific values of k and \vec{l} . The restriction of the eigenfunction to the graph vertices, however, is uniquely determined from $k\vec{\xi} \in \mathbb{T}^{|I|}$ (see e.g., [42, 43]).

We end by noting that as the entries of $\vec{\xi}$ are linearly independent over \mathbb{Q} , the flow (2.6) is ergodic on $\mathbb{T}^{|I|}$. This ergodicity is the reason for making the reduction from the set of edge lengths, $\{l_e\}_{e \in \mathcal{E}}$, to $\{\xi_i\}_{i \in I}$. We could have defined the flow on an $|\mathcal{E}|$ -dimensional torus, but then the flow would not necessarily fill the whole torus - it would actually fill a hypersurface which is $\mathcal{L}(\mathbb{T}^{|I|})$.

2.3. The spectral connection between metric and discrete graphs. An interesting connection exists between the spectra an *equilateral* metric graph, a graph whose all edge lengths are equal, and the discrete graph which shares the same connectivity. This connection is usually stated in terms of the spectrum of the *transition matrix*,

$$(2.7) \quad \mathbf{P}_{u,v} = \begin{cases} 1/d_u & (v, u) \in \overleftrightarrow{\mathcal{E}} \\ 0 & (v, u) \notin \overleftrightarrow{\mathcal{E}} \end{cases}.$$

The transition matrix is sometimes referred to as the *difference operator* or even the discrete Laplace operator, but we will not use it here to avoid confusion. One should note that the transition operator is not symmetric (and thus does not fall under the definition of the generalized Laplacian), but it is closely related to the normalized Laplacian since

$$(2.8) \quad \mathbf{P} = \mathbf{1} - \mathbf{D}^{1/2} \mathbf{L}^{(norm)} \mathbf{D}^{-1/2},$$

where $\mathbf{D} = \text{diag}\{d_v\}_{v \in \mathcal{V}}$ is a diagonal matrix which contains all vertex degrees. If λ, f are an eigenvalue and an eigenvector of $\mathbf{L}^{(norm)}$, then \mathbf{P} has $1 - \lambda$ and $\mathbf{D}^{1/2}f$ as its corresponding eigenpair. We use this connection between both operators to slightly rephrase a known result which usually refers to the spectrum of the transition matrix.

Theorem 3. [61, 19, 44, 49, 62] *Let \mathcal{G} be a discrete graph and Γ an equilateral metric graph with the same connectivity and such that $\forall e \ l_e = 1$. Consider the normalized Laplacian, $\mathbf{L}^{(norm)}$, on \mathcal{G} and the metric Laplacian with Neumann conditions on Γ . Equip both operators with magnetic fluxes for some choice of β cycles on the graph (the same choice for both \mathcal{G} and Γ). Let $\mu(\vec{\alpha}) \notin \{0, 2\}$ be an eigenvalue of \mathcal{G} and $f(\vec{\alpha})$ the corresponding eigenvector. Then*

- (1) *All values in the infinite set $\{(\arccos[1 - \mu(\vec{\alpha})])^2\}$ are eigenvalues of the magnetic metric Schrödinger operator on Γ . We consider here \arccos as a multivalued function, $\arccos : [-1, 1] \rightarrow [0, \infty)$.*
- (2) *When $\vec{\alpha} = \vec{0}$, the other eigenvalues of the metric Schrödinger operator are the Dirichlet eigenvalues, all of which belong to the set $\{(\pi n)^2\}_{n \in \mathbb{Z}}$.*
- (3) *An eigenfunction corresponding to any of the eigenvalues $(\arccos[1 - \mu(\vec{\alpha})])^2$ equals to $\mathbf{D}^{1/2}f(\vec{\alpha})$ when restricted to Γ 's vertices.*

The content of the theorem for the zero magnetic potential case appears in [44, 19, 61, 62], where they mostly treat Neumann vertex conditions (except in [62] where the so called anti-Kirchhoff conditions are treated as well). A more general derivation which includes electric and magnetic potentials as well as δ -type conditions appears in [49]. In most works above the results are stated in terms of the transition matrix, (2.7), but we prefer to relate to the normalized Laplacian as theorem 1 applies to it.

3. DISCRETE GRAPHS

The next theorem brings one of the two results which inspire this paper's title. It is stated and proved here for discrete graphs, whereas the metric case appears in the next section.

Theorem 4. *Let \mathcal{G} be a graph with discrete Schrödinger operator, all of whose eigenvalues are generic. If its nodal point count is $\forall n \ \phi_n = n - 1$ then \mathcal{G} is a tree graph.*

Proof. Assume by contradiction that \mathcal{G} is not a tree. Namely $\beta > 0$, and we may therefore supply the graph with magnetic potential and apply theorem 1. From the assumption in our theorem we get for the nodal surplus

$$\forall n \ \sigma_n = \phi_n - (n - 1) = 0.$$

We apply theorem 1 for $\lambda_n(\vec{\alpha})$ - the third part of the theorem allows to conclude that $\lambda_n(\vec{\alpha})$ has a minimum at $\vec{\alpha} = \vec{0}$ and the second part shows that this minimum is strict. Therefore the eigenvalue sum, $\sum_n \lambda_n(\vec{\alpha})$, also has a strict minimum at $\vec{\alpha} = \vec{0}$. However, since the diagonal entries of the Laplacian do not depend on the magnetic parameters, $\vec{\alpha}$, we get that $\text{trace}\{\mathbf{L}(\vec{\alpha})\}$ is a constant function. Hence we arrive at a contradiction, due to $\sum_n \lambda_n(\vec{\alpha}) = \text{trace}\{\mathbf{L}(\vec{\alpha})\}$. \square

The essence of the proof above is to show that the zero sequence is not a valid candidate as a nodal surplus sequence. This is shown by identifying $\text{trace}\{\mathbf{L}(\vec{\alpha})\}$ as a spectral invariant independent of the magnetic potential. We wish to point on similar results which are obtained from such a method. An immediate next step would be to observe that $\text{trace}\{\mathbf{L}^2(\vec{\alpha})\}$ is a

constant function of $\vec{\alpha}$ as well. Computing the Hessian of it and expressing it in terms of the eigenvalues and their Hessians, \mathcal{H}_{λ_n} , gives

$$(3.1) \quad \sum_{n=1}^{|\mathcal{V}|} \lambda_n \left(\vec{0} \right) \mathcal{H}_{\lambda_n} \left(\vec{0} \right) = \mathbf{0},$$

where we have used $\forall n \forall i \frac{\partial}{\partial \alpha_i} \lambda_n \left(\vec{0} \right) = 0$, which we get from theorem 1 (part (1)). Similarly, the magnetic invariance of trace $\{\mathbf{L}(\vec{\alpha})\}$ gives

$$(3.2) \quad \sum_{n=1}^{|\mathcal{V}|} \mathcal{H}_{\lambda_n} \left(\vec{0} \right) = \mathbf{0}.$$

Combining (3.2) and (3.1) allows to prove the following

Proposition 5. *Let \mathcal{G} be a graph with β cycles supplied with discrete Schrödinger operator such that all of its eigenvalues satisfy assumption 1. Its nodal count cannot be of the form*

$$\phi_n = \begin{cases} n - 1 & n \leq m \\ n - 1 + \beta & n > m \end{cases}$$

for any m .

Proof. Assume by contradiction that the graph has the above nodal count. Namely, the graph's surplus sequence is of the form $\{0, \dots, 0, \beta, \dots, \beta\}$. From theorem 1 we get that

$$(3.3) \quad \forall 1 \leq n \leq m \quad \mathcal{H}_{\lambda_n} \left(\vec{0} \right) > \mathbf{0}$$

$$(3.4) \quad \forall m + 1 \leq n \leq |\mathcal{V}| \quad \mathcal{H}_{\lambda_n} \left(\vec{0} \right) < \mathbf{0}$$

where the first (second) inequality above means that the Hessians are positive (negative) definite quadratic forms. If $m = 0$ we get contradiction by plugging (3.4) in (3.2). Otherwise, the contradiction comes about by the following calculation

$$\begin{aligned} \mathbf{0} &= \sum_{n=1}^m \lambda_n \left(\vec{0} \right) \mathcal{H}_{\lambda_n} \left(\vec{0} \right) + \sum_{n=m+1}^{|\mathcal{V}|} \lambda_n \left(\vec{0} \right) \mathcal{H}_{\lambda_n} \left(\vec{0} \right) \\ &< \lambda_{m+1} \left(\vec{0} \right) \sum_{n=1}^m \mathcal{H}_{\lambda_n} \left(\vec{0} \right) + \lambda_{m+1} \left(\vec{0} \right) \sum_{n=m+1}^{|\mathcal{V}|} \lambda_n \left(\vec{0} \right) \\ &= \lambda_{m+1} \left(\vec{0} \right) \sum_{n=1}^{|\mathcal{V}|} \mathcal{H}_{\lambda_n} \left(\vec{0} \right), \end{aligned}$$

where we started from (3.1) and used (3.3), (3.4) and the fact that eigenvalues are ordered increasingly to get the next line. The last line is proportional to (3.2), but $\lambda_{m+1} \left(\vec{0} \right) > 0$, which gives the contradiction. \square

Carrying on with this route and examining traces of higher powers of the operator, shows that in general trace $\{\mathbf{L}^k(\vec{\alpha})\}$ might depend on the magnetic parameters. More specifically, a direct calculation of trace $\{\mathbf{L}^k(\vec{\alpha})\}$ shows that it can be expressed as an expansion over closed walks of size k on the graph. Examining the dependence of such walks on magnetic parameters brings about the following

Theorem 6. *Let \mathcal{G} be a graph with discrete Schrödinger operator such that all of its eigenvalues satisfy assumption 1. Then the size of the graph's shortest cycle (its girth) is*

$$\min_k \left\{ \sum_i \lambda_i^{k-1} \mathcal{H}_{\lambda_i} \neq \mathbf{0} \right\}$$

Alternatively, it also equals $\min_k \{\sum_i \lambda_i^{k-1} \text{trace} \mathcal{H}_{\lambda_i} \neq 0\}$ (and $\min_k \{Z(k)\} = \infty$ is understood if there is no k for which $Z(k)$ holds).

Proof. First treat the case that \mathcal{G} is a tree. All eigenvalues are independent of the magnetic potential, and therefore all their Hessians equal zero so that both $\min_k \{\sum_i \lambda_i^{k-1} \mathcal{H}_{\lambda_i} \neq 0\}$ and $\min_k \{\sum_i \lambda_i^{k-1} \text{trace} \mathcal{H}_{\lambda_i} \neq 0\}$ equal ∞ . Indeed the girth of a tree graph equals ∞ . We now proceed to examine graphs with cycles.

The Hessian of $\text{trace} \{\mathbf{L}^k\}$ is obtained in terms of Hessians of the eigenvalues as

$$(3.5) \quad \mathcal{H}_{\text{trace}\{\mathbf{L}^k\}}(\vec{0}) = \sum_{n=1}^{|\mathcal{V}|} \lambda_n^{k-1} \mathcal{H}_{\lambda_n}(\vec{0}),$$

where we used theorem 1 to conclude that λ_n has a critical point at $\vec{\alpha} = \vec{0}$, and therefore no first derivatives appear above. The diagonal entries of \mathbf{L}^k can be expressed using closed walks on the graph. We define a *closed walk* by $\gamma = (v, v_1, \dots, v_{k-1})$, where for all $0 \leq i \leq k-1$ either $v_i = v_{i+1}$ or $(v_i, v_{i+1}) \in \overleftrightarrow{\mathcal{E}}$ (we denote $v_k := v_0 := v$ and keep in mind that γ is defined up to cyclic permutations). This is similar to the usual notion of closed walks, but we allow the walk to stop at any vertex for a (discrete) while before continuing to the next one. The set of all closed walks of size k , passing through vertex v , is denoted $\mathcal{C}_v^{(k)}$ and we attribute to each walk a weight obtained as a product of all corresponding Laplacian entries,

$$L_\gamma := \mathbf{L}_{v, v_{k-1}} \dots \mathbf{L}_{v_2, v_1} \mathbf{L}_{v_1, v}.$$

The diagonal entries of \mathbf{L}^k are given as

$$[\mathbf{L}^k]_{v,v} = \sum_{\gamma \in \mathcal{C}_v^{(k)}} L_\gamma.$$

Denote by κ the graph's girth. For $k < \kappa$, the closed walks $\gamma \in \mathcal{C}_v^{(k)}$ do not circulate any cycle and therefore their weights, L_γ , are independent of magnetic parameters (as can be also seen by direct calculation of L_γ). Therefore we get from (3.5) that $\min_k \{\sum_i \lambda_i^{k-1} \mathcal{H}_{\lambda_i} \neq 0\} \geq \kappa$. We now show that we actually have an equality. For $k = \kappa$, let $\gamma = (v, v_1, \dots, v_{k-1})$ be a closed walk on the graph which contains one of the graph's cycles. The walk γ must have all of its vertices different from each other, as k is the length of the shortest cycle on the graph. The contribution to $[\mathbf{L}^k]_{v,v}$ comes only from γ and other walks which circulates one of the graph cycles. We may couple all such walks to

$$\gamma = (v, v_1, \dots, v_{k-1}) \text{ and } \hat{\gamma} = (v, v_{k-1}, \dots, v_1),$$

and get that the contributions of these walks are complex conjugates of each other, $L_\gamma = \overline{L_{\hat{\gamma}}}$.

From $\mathbf{L}(\vec{\alpha})$ being hermitian and $\mathbf{L}(\vec{0})$ having negative off-diagonal entries, we get

$$L_\gamma + L_{\hat{\gamma}} = (-1)^k L_{(\gamma)} \cos(i\vec{n} \cdot \vec{\alpha}),$$

where $L_{(\gamma)} = |L_\gamma|$ and $\vec{n} \in \mathbb{Z}^\beta$. As γ circulates one of the graph cycles, $\vec{n} \neq \vec{0}$ and therefore

$$\exists i \text{ s.t. } \frac{\partial^2}{\partial \alpha_i^2} (L_\gamma + L_{\hat{\gamma}}) \Big|_{\vec{\alpha}=\vec{0}} \neq 0.$$

In particular, all such second derivatives which do not vanish have a definite sign, which equals to $(-1)^{k+1}$. Therefore, summing over all such couples, $\gamma, \hat{\gamma}$ gives

$$\text{sign} \left\{ \frac{\partial^2}{\partial \alpha_i^2} ([\mathbf{L}^k]_{v,v}) \Big|_{\vec{\alpha}=\vec{0}} \right\} = (-1)^{k+1},$$

and therefore also

$$\text{sign} \left\{ \frac{\partial^2}{\partial \alpha_i^2} (\text{trace} \{\mathbf{L}^k\}) \Big|_{\vec{\alpha}=\vec{0}} \right\} = (-1)^{k+1}.$$

This shows that for $k = \kappa$, the trace of the Hessian in (3.5) is different than zero and completes the proof. \square

Remark 7. Note that the proof would work similarly if the second derivatives are calculated with respect to the magnetic potential on the single edges (rather than the flux over a cycle). These derivatives are more accessible for computation, especially if the geometry of the graph is unknown, which is relevant as we are dealing with inverse problems.

Theorem 6 goes a step forward from theorem 4 in the sense that it not only distinguishes between a graph with cycles and a tree graph, but it also allows to obtain some information on the cycles. Yet, the information used in this inverse result is purely spectral - eigenvalues and their perturbations with respect to magnetic potentials. It would be interesting to see if one may obtain results of similar character from the nodal count of the graph. A possible direction might be to examine the magnetic derivatives of traces of powers of the Laplacian (as in the proof of theorem 6) or the magnetic derivatives of the coefficients of the characteristic polynomial. Theorem 1 would probably be a main tool in such an exploration. Indeed, in the course of the proof of theorem 6 we did not exploit the full strength of theorem 1 and used it only to claim that all first magnetic derivatives vanish. Applying the spectral-nodal connection which theorem 1 offers to gather information on the graph cycles, might be an important step in developing a trace formula for the nodal count. A further discussion on this direction is found in section 6.

4. METRIC GRAPHS

The main result of this paper appears as a corollary of the next theorem. The methods developed in theorem 8 and lemmata 13, 14 are aimed at proving this result. We start by introducing a convenient notation used for phrasing the results in this section

$$\mathcal{N} := \{n; \lambda_n \neq 0 \text{ and } \lambda_n \text{ satisfies assumption 1}\}.$$

The conditions and results in the section would relate to eigenvalues whose serial number are from the set \mathcal{N} . We will call such eigenvalues *generic*.

Theorem 8. *Let Γ be a Neumann metric graph with $\beta > 0$ cycles and at least one generic eigenvalue. Let $n \in \mathcal{N}$, and denote by σ the nodal surplus of the eigenvalue λ_n . Then*

- (1) *there are infinitely many generic eigenvalues whose nodal surplus equals σ .*
- (2) *there are infinitely many generic eigenvalues whose nodal surplus equals $\beta - \sigma$.*

We postpone the proof of the theorem in favour of a short discussion and two corollaries.

Corollary 9. *Let Γ be a Neumann metric graph with at least one generic eigenvalue. If $|\{n \in \mathcal{N}; \phi_n \neq n - 1 + c\}| < \infty$ then Γ has $\beta = 2c$ cycles.*

Proof. Assume that Γ has $\beta > 0$ cycles. Pick $n \in \mathcal{N}$ such that the nodal surplus of λ_n equals c . Applying theorem 8 gives that there are infinitely many eigenvalues whose nodal surplus equals $\beta - c$. This is possible if either $\beta = 0$ and theorem 8 cannot be applied or $\beta = 2c$. The second case finishes the proof. If the first case holds, then the graph is a tree and from previous works, [51, 54], we know that the nodal surplus of its generic eigenvalues equals zero. Hence it is possible only if $c = 0$ and the corollary is proved. \square

Remark 10. Note that corollary 9, with $c = 0$, is in some sense the metric analogue of theorem 4. On one hand, corollary 9 requires milder assumption than its discrete analogue - genericity assumption is required only for a single eigenvalue and not for the whole spectrum. On the other hand, corollary 9 solves the inverse problem only for the Laplacian (as it is a Neumann graph and there is no potential) whereas theorem 4 applies to the more general Schrödinger operator.

Remark 11. Phrasing somewhat loosely the $c = 0$ case of the corollary, we may say that if the nodal count of the generic eigenvalues looks 'almost' like this of a tree (up to a finite number of discrepancies), then it is a tree. As earlier works, [51, 54], show that all generic eigenvalues of a tree have nodal count, $\phi_n = n - 1$, we conclude from corollary 9 that a generic nodal count cannot be almost like this of a tree - either it equals a tree's nodal count or it differs from it for an infinite subsequence. Furthermore, it is interesting to find out whether there are graphs whose generic nodal surplus equals some constant $c > 0$ (maybe up to a finite number of discrepancies). This will reveal whether or not corollary 9 is empty for the $c > 0$ case. We emphasize once again that all said above relates only to generic eigenvalues. We did not consider the non generic ones, as we did not establish in this paper a well defined way for counting the zeros in such cases (and problems indeed arise either from the multiplicity of the eigenvalue or from zeros at the vertices of eigenfunctions).

Corollary 9 allows to prove an analogue of itself for the nodal domain count

Corollary 12. *Let Γ be a Neumann metric graph with at least one generic eigenvalue. If $|\{n \in \mathcal{N}; \nu_n \neq n - c\}| < \infty$ then Γ has $\beta = 2c$ cycles.*

Proof. We use the following connection between nodal point count and nodal domain count for metric graphs, which is proved for example in [4] (see there, equation (1.11) together with lemma 3.2),

$$(4.1) \quad \exists n_0 \text{ s.t. } \forall n > n_0 \quad \nu_n = \phi_n + 1 - \beta.$$

This connection together with the corollary assumption gives $|\{n \in \mathcal{N}; \phi_n \neq n - 1 + \beta - c\}| < \infty$. From corollary 9 we may now deduce that the graph has $2(\beta - c)$ cycles. Namely, $\beta = 2(\beta - c)$, from which we get $\beta = 2c$. \square

Having discussed the main results which stem from theorem 8, we now state and prove two lemmata which form the main tool in the theorem's proof, which follows them. The following results are stated in terms of the graph's k -spectrum, $k_n(\vec{\alpha}) = \sqrt{\lambda_n(\vec{\alpha})}$, for the sake of simplicity.

Lemma 13. *Let $\vec{\xi} \in \mathbb{R}^I$ be a vector of incommensurate entries which express the graph edge lengths by (2.2). Let $k(\vec{0})$ be a generic eigenvalue. The Hessian of this eigenvalue with respect to magnetic fluxes at $\vec{\alpha} = \vec{0}$ is given as*

$$(4.2) \quad \mathcal{H}_k(\vec{0}) = \frac{\mathcal{H}_F}{\left(\vec{\xi} \cdot \vec{\nabla} F\right)} \Big|_{(k(\vec{0})\vec{\xi}; \vec{0})},$$

where \mathcal{H}_F denotes the Hessian of the secular function $F(k(\vec{0})\vec{\xi}; \cdot)$ with respect to its magnetic parameters and $\vec{\nabla} F$ is the gradient of $F(\cdot; \vec{0})$ taken with respect to its coordinates on the torus $\mathbb{T}^{|I|}$.

Proof. The eigenvalue $k(\vec{\alpha})$ is given implicitly as the solution of $F(k(\vec{\alpha})\vec{\xi}; \vec{\alpha}) = 0$. Take second total derivatives of this with respect to any two magnetic fluxes,

$$0 = \frac{d^2}{d\alpha_i d\alpha_j} F(k(\vec{\alpha})\vec{\xi}; \vec{\alpha}) = \left(\vec{\xi} \cdot \vec{\nabla} F\right) \frac{\partial^2 k}{\partial \alpha_i \partial \alpha_j} + \left(\vec{\xi} \cdot \frac{\partial}{\partial \alpha_j} \vec{\nabla} F\right) \frac{\partial k}{\partial \alpha_j} + \frac{\partial^2 F}{\partial \alpha_i \partial \alpha_j}.$$

Evaluate the above at the point $\vec{\alpha} = \vec{0}$ and use $\frac{\partial k}{\partial \alpha_j} \Big|_{\vec{\alpha}=\vec{0}} = 0$, which we obtain as an application of theorem 1 (part (1)), to get

$$(4.3) \quad \left\{ \left(\vec{\xi} \cdot \vec{\nabla} F\right) \mathcal{H}_k(\vec{0}) + \mathcal{H}_F \right\} \Big|_{(k(\vec{0})\vec{\xi}; \vec{0})} = 0.$$

Consider the zero set of $F(\cdot; \vec{0})$, $\Sigma_{\vec{0}} = \{\vec{x} \in \mathbb{T}^{|I|}; F(\vec{x}; \vec{0}) = 0\}$. We assumed that the eigenvalue $k(\vec{0})$ is simple and therefore the hyperplanes of which $\Sigma_{\vec{0}}$ consists $F(\cdot; \vec{0})$ do not cross at the point $k(\vec{0})\vec{\xi}$. This means that $\vec{\nabla}F|_{(k(\vec{0})\vec{\xi}, \vec{0})} \neq \vec{0}$. Furthermore, we now explain why $\vec{\xi} \cdot \vec{\nabla}F|_{(k(\vec{0})\vec{\xi}, \vec{0})} \neq \vec{0}$ as well. If the product $\vec{\xi} \cdot \vec{\nabla}F|_{(k(\vec{0})\vec{\xi}, \vec{0})}$ was equal to zero equation (4.3) would imply $\mathcal{H}_F = \mathbf{0}$. One may then choose $\vec{\omega}$ such that $\vec{\omega} \cdot \vec{\nabla}F|_{(k(\vec{0})\vec{\xi}, \vec{0})} \neq \vec{0}$ and $\tilde{k} \cdot \vec{\omega} = k(\vec{0})\vec{\xi}$ modulus $\mathbb{T}^{|I|}$. The vector $\vec{\omega}$ can be used to define a different flow on the torus $\mathbb{T}^{|I|}$, which pierces $\Sigma_{\vec{0}}$ at the same point, $\tilde{k} \cdot \vec{\omega} = k(\vec{0})\vec{\xi}$, and yields an eigenvalue \tilde{k} . Equation (4.3) (applied for the new flow) together with $\mathcal{H}_F = \mathbf{0}$ then gives $\mathcal{H}_{\tilde{k}} = \mathbf{0}$ which contradicts the application of theorem 1 (part (2)) to \tilde{k} and finally shows that $\vec{\xi} \cdot \vec{\nabla}F|_{(k(\vec{0})\vec{\xi}, \vec{0})} \neq \vec{0}$. We may therefore divide the equation above by $\vec{\xi} \cdot \vec{\nabla}F|_{(k(\vec{0})\vec{\xi}, \vec{0})}$ to get (4.2). \square

Lemma 14. *The secular function F of a Neumann graph exhibits the following symmetry*

$$(4.4) \quad F(\vec{x}; \vec{\alpha}) = F(-\vec{x}; -\vec{\alpha}).$$

Proof. From the definition of the secular function, \tilde{F} , in (2.1) we get

$$\tilde{F}(k; \vec{l}; \vec{\alpha}) = \overline{\tilde{F}(-k; \vec{l}; -\vec{\alpha})} = \tilde{F}(-k; \vec{l}; -\vec{\alpha}).$$

The first equality can be deduced from (2.1) together with the fact that $\mathbf{S}(k)$ is real and k -independent for a Neumann graph and the second one is due to \tilde{F} being real. Now combine this with property (2) mentioned after the definition of F on the torus, (2.5), to get (4.4). \square

Proof. [Proof of theorem 8] Denote the edge lengths of Γ by the vector \vec{l} and choose an incommensurate set $\vec{\xi} \in \mathbb{R}^I$ which is related to graph edge lengths by (2.2). Let k be a generic eigenvalue of Γ whose nodal surplus is σ . The Hessian of k is given by relation (4.2) in lemma

13. As $\frac{\mathcal{H}_F}{(\vec{\xi} \cdot \vec{\nabla}F)}|_{(k\vec{\xi}, \vec{0})}$ is non degenerate, we may choose a neighbourhood Ξ of $k\vec{\xi}$ on $\Sigma_{\vec{0}}$ such

that $\frac{\mathcal{H}_F}{(\vec{\xi} \cdot \vec{\nabla}F)}|_{(\vec{x}, \vec{0})}$ is non degenerate for all $\vec{x} \in \Xi$, and therefore its Morse index is constant in

this neighbourhood. This Morse index equals to the Morse index of the eigenvalue and it equals to σ by an application of theorem 1 (part (3)). As the flow is ergodic, the set $\Xi \subset \Sigma_{\vec{0}}$ will be pierced an infinite number of times by the flow, yielding infinitely many eigenvalues. These eigenvalues are simple since Ξ can be chosen not to contain intersections of $\Sigma_{\vec{0}}$'s hyperplanes. In addition, we may choose Ξ such that the corresponding eigenfunctions do not vanish at graph vertices (see remark 2). Therefore, all eigenvalues obtained by the flow piercing Ξ are generic. Each of these eigenvalues would have this same Morse index, σ , by (4.2). The nodal surplus of each of these eigenvalues is therefore also equal to σ , according to theorem 1 (part (3)) and this proves the first statement in our theorem. The second statement is proved with the aid of the symmetry (4.4) in lemma 14, which allows to conclude

$$\begin{aligned} \forall \vec{x} \in \Xi \quad ; \quad \mathcal{H}_F|_{(\vec{x}, \vec{0})} &= \mathcal{H}_F|_{(-\vec{x}, \vec{0})} \\ \vec{\nabla}F|_{(\vec{x}, \vec{0})} &= -\vec{\nabla}F|_{(-\vec{x}, \vec{0})}, \end{aligned}$$

form which we get

$$(4.5) \quad \forall \vec{x} \in \Xi ; \left. \frac{\mathcal{H}_F}{(\vec{\xi} \cdot \vec{\nabla} F)} \right|_{(\vec{x}, \vec{0})} = - \left. \frac{\mathcal{H}_F}{(\vec{\xi} \cdot \vec{\nabla} F)} \right|_{(-\vec{x}, \vec{0})}.$$

The Morse index of the LHS equals σ , which gives that the Morse index of the RHS is $\beta - \sigma$. We now use again the ergodicity of the flow to conclude that the set $-\Xi = \{\vec{x}; -\vec{x} \in \Xi\} \subset \Sigma_{\vec{0}}$ will be pierced an infinite number of times by the flow and all resulting eigenvalues would have Morse index $\beta - \sigma$ and this would be also their nodal surplus following theorem 1. \square

Remark 15. This proof is of a different nature than the proof of the discrete inverse nodal theorem (theorem 4), where we have identified the trace of the operator as a spectral invariant independent of magnetic potential. One can find, however, a similarity between the proofs, as the antisymmetric relation (4.5) points on the possibility to average the Hessians over whole torus and show that the magnetic dependence cancels.

Remark 16. In the course of the proof we have used an equality between the Morse index of a particular eigenvalue and the Morse index of the secular function F evaluated at the appropriate point. A similar relation appears in appendix E of [26] where the Morse index of the characteristic polynomial is compared to the Morse index of the particular eigenvalue for the case of a discrete graph.

Remark 17. The symmetry on which lemma 14 points can be already exhibited on the level of the Schrödinger operator on the graph. One can show that the $\vec{\alpha} \rightleftharpoons -\vec{\alpha}$ symmetry amounts to conjugation of all eigenvalues, which actually leaves the spectrum invariant as it is real. This symmetry was exploited in [26, 14] to prove that the eigenvalues have critical points at $\vec{\alpha} = \vec{0}$.

5. DISCRETIZED VERSIONS OF A METRIC GRAPH

This section describes a construction which fits a given metric graph various discrete graphs whose surplus sequence resembles this of the metric graph in a sense described below. This correspondence between a metric graph and its discretized versions is interesting on its own and can also be used as a tool to provide an additional proof to the inverse result in corollary 9 and to further extend it for graphs with general vertex conditions and electric potentials.

The construction starts by picking a vector $\vec{\xi} \in \mathbb{R}^I$ of incommensurate entries such that the graph edge lengths are given by $\vec{l} = \mathcal{L}(\vec{\xi})$ and \mathcal{L} is a linear map over \mathbb{Q} (see section 2.2). Our construction is characterized by a selection of

$$(5.1) \quad \vec{j} \in \text{Image}(\mathcal{L}) \cap \mathbb{N}^{\mathcal{E}}.$$

Note that the set $\text{Image}(\mathcal{L}) \cap \mathbb{N}^{\mathcal{E}}$ is non empty. In order to show this, one can approximate $\vec{\xi}$ by some rational vector, $\vec{\xi}^{\text{rat}} \in \mathbb{Q}^I$, such that $\mathcal{L}(\vec{\xi}^{\text{rat}})$ has all positive entries (just as $\vec{l} = \mathcal{L}(\vec{\xi})$ does). Note that $\mathcal{L}(\vec{\xi}^{\text{rat}}) \in \mathbb{Q}^{\mathcal{E}}$ (see (2.2)), and therefore the vector $\mathcal{L}(\vec{\xi}^{\text{rat}})$ can now be multiplied by a common divisor of its entries to turn its entries to natural numbers, while retaining it in $\text{Image}(\mathcal{L})$.

Equip each edge $e \in \mathcal{E}$ with $j_e - 1$ new vertices of degree two, which would split this edge into j_e new edges. Now, consider a discrete graph which inherits from this (new) metric graph its sets of vertices and edges and their connectivity. This discrete graph is denoted \mathcal{G}_{Γ} and called *a discretized version of Γ* . Note that a discretized version is not uniquely determined by Γ . The set of all possible discretized versions is given by $\text{Image}(\mathcal{L}) \cap \mathbb{N}^{\mathcal{E}}$. This set depends on the 'nature of incommensurability' of the original edge lengths, $\{l_e\}_{e \in \mathcal{E}}$, i.e., what are the rational dependencies between the lengths. However, one can verify that $\text{Image}(\mathcal{L}) \cap \mathbb{N}^{\mathcal{E}}$ does not depend on the particular choice of incommensurate representatives, $\{\xi_i\}_{i \in I}$.

Note that one may convert the discretized graph, \mathcal{G}_Γ , back into a metric graph (different than Γ) by setting all of \mathcal{G}_Γ 's edge lengths to equal one. One would then get a metric graph with the same connectivity as Γ , but with integer edge lengths given by \vec{j} . We denote this equilateral version of the discretized graph by $\tilde{\Gamma}$ and it will turn to be useful in the course of proving the following

Theorem 18. *Let Γ be a Neumann metric graph with $\beta > 0$ cycles and let \mathcal{G}_Γ be a discretized version of Γ . Let $\mu \notin \{0, 2\}$ be an eigenvalue of $\mathbf{L}^{(norm)}$ on \mathcal{G}_Γ which satisfies assumption 1 and whose nodal surplus is $\sigma^{(\mathcal{G}_\Gamma)}$. Then*

- (1) Γ has infinitely many generic eigenvalues whose nodal surplus equals $\sigma^{(\mathcal{G}_\Gamma)}$.
- (2) Γ has infinitely many generic eigenvalues whose nodal surplus equals $\beta - \sigma^{(\mathcal{G}_\Gamma)}$.

Remark 19. The theorem is empty if the spectrum of the discretized version consists only of the eigenvalues $\{0, 2\}$ and other eigenvalues which do not satisfy assumption 1. The only connected graph which has no eigenvalues different from $\{0, 2\}$ is the single edge graph (see for example, lemma 1.8 in [20]). Nevertheless it does not fit our theorem as it has no cycles. There are, however, discrete graphs with cycles, whose all eigenvalues which are different than $\{0, 2\}$ are not simple. The n -cube graph forms such an example (example 1.6 in [20]). The question which remains is whether for a given metric graph, there is always some discretized version for which theorem 18 is not empty.

Remark 20. This theorem is ought to be compared with theorem 8, as their conclusions are similar. Yet, the current theorem seems weaker as it requires more conditions (see previous remark for example). The proof, however, contains an element which allows to bypass the need of symmetry (lemma 14) which is required in the proof of theorem 8 and thus gives the possibility to generalize this result to non Neumann graphs and graphs with potentials.

Proof. Start by proving the theorem for an equilateral graph Γ , and a specific discretization of it. We choose a discretized version, \mathcal{G}_Γ , which has the same vertex and edge sets and connectivity as Γ does (it is obtained by choosing $\vec{j} = (1, \dots, 1)$ in (5.1)). Let μ be an eigenvalue of $\mathbf{L}^{(norm)}(\vec{0})$ on \mathcal{G}_Γ which satisfies assumption 1 and whose eigenvector is f and nodal surplus is $\sigma^{(\mathcal{G}_\Gamma)}$. Consider the magnetic Laplacian, $\mathbf{L}^{(norm)}(\vec{\alpha})$, apply theorem 1 and obtain that $\mu(\vec{\alpha})$ has a critical point at $\vec{\alpha} = \vec{0}$ and its Morse index is $\mathcal{M}_\mu(\vec{0}) = \sigma^{(\mathcal{G}_\Gamma)}$. Theorem 3 shows that $\{(\arccos[1 - \mu(\vec{\alpha})])^2\}$ are eigenvalues of the magnetic Schrödinger operator on Γ and we wish to obtain their Morse indices at $\vec{\alpha} = \vec{0}$. First, exclude the values ± 1 from the domain of arccos and consider it as a union of single valued 'branch' functions $\{b_p : (-1, 1) \rightarrow (p\pi, (p+1)\pi)\}_{p=0}^\infty$. With this notation, those eigenvalues of Γ resulting from $\mu(\vec{\alpha})$ are given by

$$\lambda_{(p)}(\vec{\alpha}) = (b_p[1 - \mu(\vec{\alpha})])^2,$$

where the subscript $_{(p)}$ is not to be confused with the serial number of $\lambda_{(p)}$ in Γ 's spectrum. Each of those eigenvalues is obtained as a function of $\mu(\vec{\alpha})$, which has a well defined monotonicity:

- (1) For even values of p , $\lambda_{(p)}(\mu(\vec{\alpha}))$ is a monotone strictly increasing function of $\mu(\vec{\alpha})$. Therefore, $\lambda_{(p)}(\vec{\alpha})$ has a critical point at $\vec{\alpha} = \vec{0}$, as well as $\mu(\vec{\alpha})$, and their Hessians are equal up to a positive multiple, which yields equality of their Morse indices,

$$(5.2) \quad \mathcal{M}_{\lambda_{(p)}}(\vec{0}) = \mathcal{M}_\mu(\vec{0}).$$

- (2) For odd values of p , $\lambda_{(p)}(\mu(\vec{\alpha}))$ is a monotone strictly decreasing function of $\mu(\vec{\alpha})$. This time, the Hessians of $\lambda_{(p)}$ and μ are equal up to a negative multiple, which yields the following relation of their Morse indices

$$(5.3) \quad \mathcal{M}_{\lambda_{(p)}}(\vec{0}) = \beta - \mathcal{M}_\mu(\vec{0}).$$

We may now apply the metric version of theorem 1 for the eigenvalues $\lambda_{(p)}$, but first verify that they satisfy the theorem's assumptions. Their simplicity (at $\vec{\alpha} = \vec{0}$) follows from theorem 3 as $\mu(\vec{0})$ is simple and $\mu(\vec{0}) \notin \{0, 2\}$ (which guarantees that $\lambda_{(p)}(\vec{0}) \notin \{(n\pi)^2\}_{n \in \mathbb{Z}}$, i.e. different from the Dirichlet eigenvalues). In addition, according to theorem 3 (part (3)), the restriction of the eigenfunction of $\lambda_{(p)}(\vec{0})$ to the graph vertices equals $\mathbf{D}^{1/2}f$ and f , the eigenvector corresponding to $\mu(\vec{0})$, is different than zero on all vertices, by the theorem's assumption. The nodal surplus of $\lambda_{(p)}$, which we denote by $\sigma^{(\Gamma)}(\lambda_{(p)})$ can therefore be expressed as

$$(5.4) \quad \text{for even } p, \quad \sigma^{(\Gamma)}(\lambda_{(p)}) = \mathcal{M}_{\lambda_{(p)}}(\vec{0}) = \mathcal{M}_{\mu(\vec{\alpha})}(\vec{0}) = \sigma^{(\mathcal{G}_\Gamma)}$$

$$(5.5) \quad \text{for odd } p, \quad \sigma^{(\Gamma)}(\lambda_{(p)}) = \mathcal{M}_{\lambda_{(p)}}(\vec{0}) = \beta - \mathcal{M}_{\mu(\vec{\alpha})}(\vec{0}) = \beta - \sigma^{(\mathcal{G}_\Gamma)},$$

which proves the theorem for the case of an equilateral metric graph if its discretization given by choosing $\vec{j} = (1, \dots, 1)$.

If the graph Γ is not equilateral and we choose an arbitrary discretized version, \mathcal{G}_Γ , there is no exact expression which connects both spectra of Γ and \mathcal{G}_Γ . The route we take this time is to consider another metric graph, $\tilde{\Gamma}$, which has the connectivity of \mathcal{G}_Γ and all edge lengths equal to one. Note that $\tilde{\Gamma}$ is an equilateral graph and \mathcal{G}_Γ can be also considered as its discretization. There are Morse index connections similar to (5.2), (5.3) between $\tilde{\Gamma}$ and \mathcal{G}_Γ . We then show that infinitely many eigenvalues of Γ share the same Morse index as eigenvalues of $\tilde{\Gamma}$, due to the ergodic flow on the torus and this yields the desired statements in the theorem. This is the content of the rest of the proof.

Let $\mu \notin \{0, 2\}$ be an eigenvalue of \mathcal{G}_Γ which satisfies assumption 1. We may therefore conclude, just as in the first part of the proof, that $\mathcal{H}_\mu(\vec{0})$ equals up to a factor the Hessians of infinitely many eigenvalues of $\tilde{\Gamma}$ (the factors are both positive negative factors with equal proportions). Denote by \vec{j} the vector which generates the discretization \mathcal{G}_Γ . Consider $\tilde{\Gamma}$ as a metric graph with the same connectivity as Γ , but with $\{j_e\}_{e \in \mathcal{E}}$ as the set of its edge lengths. The k -eigenvalues of $\tilde{\Gamma}$ are then described by

$$\left\{ k(\vec{\alpha}); \quad k\vec{j} \in \Sigma_{\vec{\alpha}} \right\},$$

where $\Sigma_{\vec{\alpha}} = \{\vec{x}; F(\vec{x}; \vec{\alpha}) = 0\}$ is a union of hyperplanes on the torus $\mathbb{T}^{|I|}$. In particular, we know from theorem 3 that $\tilde{\Gamma}$ has the k -eigenvalues $\{f_p[1 - \mu(\vec{\alpha})]\}_{p \in \mathbb{N} \cup \{0\}}$. Choose some two k -eigenvalues with different parity of p , for example

$$\begin{aligned} k_0(\vec{\alpha}) &:= f_0[1 - \mu(\vec{\alpha})] \\ k_1(\vec{\alpha}) &:= f_1[1 - \mu(\vec{\alpha})]. \end{aligned}$$

From a similar monotonicity argument, as in the first part of the proof, we get that

$$(5.6) \quad \mathcal{H}_{k_0}(\vec{0}) = -c\mathcal{H}_{k_1}(\vec{0}),$$

where $c > 0$ and these Hessians are non degenerate. Use relation (4.2) in lemma 13 to write

$$\mathcal{H}_{k_i}(\vec{0}) = \frac{\mathcal{H}_F}{(\vec{j} \cdot \vec{\nabla} F)} \Big|_{(k_i(\vec{0})\vec{j}; \vec{0})}, \quad (i = 0, 1).$$

As these Hessians are non degenerate, we may choose neighbourhoods Ξ_i ($i = 0, 1$) of $k_i(\vec{0}) \cdot \vec{j}$ such that $\forall \vec{x} \in \Xi_i, \frac{\mathcal{H}_F}{(\vec{j} \cdot \vec{\nabla} F)} \Big|_{(\vec{x}; \vec{0})}$ is non degenerate and we may define the corresponding Morse

index function

$$(5.7) \quad \begin{aligned} m_{\vec{j}} : (\Xi_0 \cup \Xi_1) &\rightarrow \mathbb{N} \cup \{0\} \\ m_{\vec{j}}(\vec{x}) &:= \mathcal{M} \left(\frac{\mathcal{H}_F}{(\vec{j} \cdot \vec{\nabla} F)} \Big|_{(\vec{x}; \vec{0})} \right). \end{aligned}$$

Note that $m_{\vec{j}}$ is constant on each set Ξ_i , due to non degeneracy of the Hessians. From (5.6) we get the following relation on these Hessians

$$\forall \vec{x}_0 \in \Xi_0, \vec{x}_1 \in \Xi_1 \quad m_{\vec{j}}(\vec{x}_0) + m_{\vec{j}}(\vec{x}_1) = \beta.$$

Let us now return to the original graph, Γ . The flow which characterizes its eigenvalues is given by $\vec{\xi}$ and we wish to adapt the definition of the Morse index function, (5.7), to this flow. We now show that $\vec{j} \cdot \vec{\nabla} F|_{(\vec{x}; \vec{0})}$ and $\vec{\xi} \cdot \vec{\nabla} F|_{(\vec{x}; \vec{0})}$ have the same sign on Ξ_i and conclude that

$$m_{\vec{\xi}}|_{\Xi_0 \cup \Xi_1} \equiv m_{\vec{j}}|_{\Xi_0 \cup \Xi_1}.$$

We have shown in the proof of lemma 13 that $\vec{j} \cdot \vec{\nabla} F|_{(k_i(\vec{0}); \vec{0})}$ cannot vanish. We may therefore assume that $\vec{j} \cdot \vec{\nabla} F|_{(\vec{x}; \vec{0})}$ does not vanish on the sets Ξ_i (otherwise, restrict Ξ_i to a smaller set where this holds). The argument used in lemma 13 can be used to show that $\forall t \in [0, 1], \forall \vec{x} \in (\Xi_0 \cup \Xi_1), (t\vec{j} + (1-t)\vec{\xi}) \cdot \vec{\nabla} F|_{(\vec{x}; \vec{0})} \neq 0$ which shows that $\vec{j} \cdot \vec{\nabla} F|_{(\vec{x}; \vec{0})}$ and $\vec{\xi} \cdot \vec{\nabla} F|_{(\vec{x}; \vec{0})}$ have the same sign. As the flow defined by $\vec{\xi}$ pierces both Ξ_0 and Ξ_1 an infinite number of times, we get infinite number of eigenvalues of Γ whose Morse indices are $\sigma^{(\mathcal{G}_\Gamma)}$ and $\beta - \sigma^{(\mathcal{G}_\Gamma)}$. Applying theorem 1 finishes the proof. \square

Remark 21. The structure of the last proof might allow a generalization for non Neumann graphs. It was shown in [49] that theorem 3 holds also for δ -type conditions and for some electric potentials, if the the inverse Hill discriminant is used instead of the arccos. The mechanism of ergodic flow on the torus does not describe accurately the eigenvalues of δ -type conditions, but it does so asymptotically, which should be enough for our purpose. The case of electric potential might be treated as well, as it was shown in recent work [53] that its spectrum can be described asymptotically by secular function similar to (2.1).

6. A DISCUSSION

The main result of this paper, as is implied by the title, is the solution of the inverse nodal problem of determining a tree graph. The solution is similar for both metric and discrete graphs - the nodal point count sequence $\{0, 1, 2, 3, \dots\}$ may be possessed solely by tree graphs, under some genericity assumptions. The similarity between discrete and metric graphs carries over to the proofs - both use as a crucial tool the recently established connection between the graph's nodal count and dependence of its eigenvalues on magnetic fields, [11, 14, 26]. The proof for discrete graphs is based on a very basic observation - the trace of the operator does not depend on magnetic fields. The proof for the metric case is of more exploratory type and concerns properties of some individual eigenvalues. This proof yields some additional results and offers further investigative directions. Yet, it might seem superfluous for our main purpose. It would be interesting to develop an alternative proof for the metric inverse problem which tackles a specific spectral invariant similarly to the discrete case. Possible candidates which arise as natural generalizations of the trace are the vacuum energy with some regularization, or the value of the zeta function at some point. Having said that, one should note that the proof of theorem 8 does include an implicit spectral invariant (see remark 15). This work also sets some

restrictions on possible nodal count sequences which one can obtain from a graph. For example, we show that non tree graphs cannot have a nodal count sequence which is almost like the tree nodal count (up to a finite number of discrepancies). In this sense, our result resembles a recent 'quasi-isospectrality' result by Rueckriemen, [52]. He shows that if the spectra of two graphs agree everywhere aside from a sufficiently sparse set, then they are isospectral. In both his and our case, there are typical sequences (either nodal or spectral) which characterize the graphs and do not allow for 'small' number of discrepancies.

Putting aside the connection to the nodal count, one could phrase the results we obtained as purely magnetic properties of graphs' spectra. It is not possible for all graph eigenvalues to show diamagnetic behaviour (see chapter 2 in [29]). This statement holds for discrete graphs, whereas metric graphs obey even a stronger restriction - an infinite number of eigenvalues must violate the diamagnetic behaviour.

The discrete part of the inverse theorem was proved with quite a high generality by allowing a dense set of discrete Schrödinger operators. However, in the metric case we permit variable edge lengths, but restrict ourselves to Neumann vertex conditions and with no electric potentials. It is desirable to generalize the current results to include other vertex conditions and potentials. A first step for doing so is suggested by the method of discretized versions of a graph presented in section 5. See also remark 21 which offers a possible approach for such generalizations. Of particular interest is the question whether for a given metric graph, there is always some discretized version for which theorem 18 is not empty.

The inverse nodal domain count problem for metric graphs was solved as well in this paper, i.e., the nodal domain count sequence, $\{1, 2, 3, \dots\}$ implies the graph is a tree. Numerical evidence suggests that this should be the case for discrete graphs as well and it is interesting to prove (or maybe disprove) such an inverse result. A possible approach might be to check if a similar magnetic-nodal connection exist for the nodal domain count as well (which would form a non-trivial generalization of the works [26, 11, 14]) and to apply it for the inverse problem. Another generalization direction of the magnetic-nodal link, from which inverse problems would benefit, is to give some treatment for non simple eigenvalues and for eigenfunctions which vanish at vertices. This is highly relevant, as the frequently used discrete standard Laplacian tend to have such non generic spectra.

Let us consider the inverse results in the paper from a spectral geometric viewpoint. We have managed to distinguish a graph with cycles from a tree by means of the nodal count. The next almost immediate problem would be to try and deduce the exact number of cycles out of the nodal count, if this is possible at all. We know that under generic assumptions, the number of cycles, β , is the upper bound of the nodal surplus. One would then inquire whether this bound is attained somewhere in the spectrum, i.e. whether

$$(6.1) \quad \beta = \max_n \{\phi_n - (n - 1)\}?$$

In the discrete graphs setting, this is not the general case as shows the following simple counter example

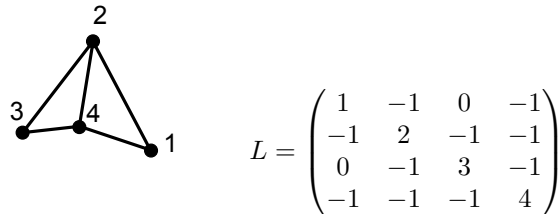


FIGURE 6.1. A simple graph with a choice of its Laplacian

The graph in figure 6.1(a) has $\beta = 2$, yet its nodal point count is $\{0, 2, 3, 3\}$ (with respect to the given Laplacian matrix) and therefore $\max_n \{\phi_n - (n - 1)\} = 1$. Note however, that the nodal

count depends not only on the graph, but also on the chosen Laplacian for it (within the possible models described in section 1). For example, had we replaced the diagonal of the Laplacian with $\{4, 3, 2, 1\}$ we would have gotten the nodal count $\{0, 3, 3, 4\}$, which does attain the maximum in (6.1). We may therefore still wonder whether statement (6.1) holds within a certain class of Laplacians. If we consider the normalized Laplacian on the same graph, for example, we get that the eigenvalues are simple, but some eigenfunctions vanish at graph vertices and anyway statement (6.1) fails for the generic eigenvalues. The same is true for the so-called standard Laplacian as well (same Laplacian as above, but with the vertex degrees on its diagonal), where even not all eigenvalues are simple. This calls for appropriate definitions of the nodal count for non generic eigenvalues and possible generalizations of theorem 1. The question of deducing β from the nodal count can be also asked for the metric graph setting. We show that indeed under a genericity assumption one may deduce the number of graph cycles out of the nodal count. This is to be discussed in a future work, where the nodal surplus distribution of a metric graph is studied, [3]. One should compare this inverse nodal problem to its analogous spectral one - whether the spectrum reveals the number of graph cycles. This is indeed the case, both for discrete graphs (see e.g., lemma 4 in [60]) and for metric graphs ([45]). This comparison between the spectral data and the nodal one goes hand to hand with the aforementioned conjecture in [34] that nodal information resolves spectral ambiguity whenever it occurs. In the discrete graph example shown above we, however, have the information flow in the opposite way than the one of the conjecture - the spectrum stores some geometric information which the nodal count does not reveal. Yet, for metric graphs, as was mentioned here, both spectral and nodal sequences tell the number of graph cycles. Finally, we wish to mention trace formulae, which are a major tool for treating inverse problems, as they connect spectral information to geometry. For graphs, these formulae express spectral functions in terms of closed paths on the graph. A trace formula for the spectral counting function of a metric graph, [42], was used to solve the inverse spectral problem on metric graphs [35]. It was suggested by Smilansky that similar trace formulae exist for functions of the nodal count and there is indeed some supporting evidence and derivations for specific classes of two dimensional surfaces in [1, 2, 32], and a progress on the problem for metric graphs in [5]. Yet, an exact trace formula for the nodal count has not been found yet. Such a formula will crucially advance solutions of inverse nodal problems of the kind presented in this paper and of many more.

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